Lecture 1
Mathematical Prelimiaries 1

We consider a model which there are plural possible outcomes. Let Ω be the set of all possible outcomes. Note that Ω is called the sample space. To simplify argument, we assume, throughout the course, that Ω is finite. Now, we denote $\Omega = \{x_1, x_2, \ldots, x_N\}$, that is, Ω is composed of N elements.

Example 1 (Tossing a coin) N in this model is given by 2, and Ω should be denoted by $\{H, T\}$. H and T means "heads" and "tails", respectively.

Example 2 (Rolling a die) In this model, N is given by 6, and Ω is denoted by $\{x_1, x_2, \ldots, x_6\}$. Each ω_i, $i = 1, 2, \ldots, 6$ is corresponding to the event "the spot i appears".

Random Variable

A random variable is a function on Ω. Usually, a random variable is $\mathbf{R}$-valued. Let X be a random variable on $\Omega = \{x_1, x_2, \ldots, x_N\}$. When we fix an $\omega \in \Omega$, $X(\omega)$ values in a real number.

Example 3 (Rolling a die) Let Y be a random variable which represents the outcome for a roll of a die. For example, ω_3 represents an event that the outcome is the spot 3. Hence, $Y(\omega_3) = 3$. Moreover, we have $Y(\omega_i) = i$ for $i = 1, 2, \ldots, 6$.

Example 4 (Rolling a die) We consider a game as follows: Firstly, we roll a die. We obtain then points which equal to the square of its outcome.

Let Z be a random variable which represents the points of the game. For example, ω_3 represents an event that the outcome is the spot 3. Thus, $Z(\omega_i) = Y(\omega_i)^2 = i^2$ for $i = 1, 2, \ldots, 6$.

Probability

Let $\mathcal{F}$ be the family of all subset of Ω, which is called the power set of Ω. Remark that each element of $\mathcal{F}$ is also a set. ($\mathcal{F}$ is a set of sets.)

P is a probability, if P is a $[0, 1]$-valued function on $\mathcal{F}$ satisfying
(1) $P(\Omega) = 1$,

(2) $P(A) \geq 0$ for any $A \in \mathcal{F}$
(3) $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$, where $\emptyset$ represents the emptyset.

Example 5 (Rolling a die) Letting A be $\{\omega_1, \omega_3, \omega_5\}$, $P(A) = 1/2$. $P(A)$ is the probability that the outcome is an odd number. Letting B be $\{\omega_5\}$, $P(B) = P(\{\omega_5\}) = 1/6$. Remark that ω is not equivalent to $\{\omega\}$. $\{\omega\}$, $\{\omega_1, \omega_2\}$, Ω, $\emptyset$ and so on are sets. On the other hand, ω is not a set, merely an element of Ω. Finally, $P(\{\omega | X \geq 4\})(=: P(X \geq 4)) = 1/2$.

Expectation

Let X be a radom variable, and $\Omega = \{x_1, x_2, \ldots, x_N\}$ be given. The expectation of X is denoted by $E[X]$.

The expectation is the average value. Mathematically, the expectation of a random variable is given by a real number, which is defined as

$$E[X] := \sum_{i=1}^{N} X(\omega_i) P(\{\omega_i\}).$$

Example 6 (Rolling a die) Let Y and Z be the random variables defined in Examples 3 and 4. We have then

$$E[Y] = \sum_{i=1}^{6} Y(\omega_i) P(\{\omega_i\}) = \sum_{i=1}^{6} i \frac{1}{6} = \frac{7}{2},$$

and

$$E[Z] = \sum_{i=1}^{6} Z(\omega_i) P(\{\omega_i\}) = \sum_{i=1}^{6} i^2 \frac{1}{6}.$$