Lecture 2

Mathematical Prelimiaries 2

Some properties of expectation

1. Linearlity

$$E[\alpha X] = \alpha E[X],$$

where $\alpha \in \mathbf{R}$ and X is a random variable.

Proof.

$$E[\alpha X] = \sum_{i=1}^{N} \alpha X(\omega_i) P(\{\omega_i\}) = \alpha E[X].$$

□

$$E[X+Y] = E[X] + E[Y],$$

where X and Y are random variables.

Proof.

$$\begin{aligned} E[X+Y] &= \sum_{i=1}^{N} \left(X(\omega_i) + Y(\omega_i) \right) P(\{\omega_i\}) \\ &= \sum_{i=1}^{N} \left\{ X(\omega_i)P(\{\omega_i\}) + Y(\omega_i)P(\{\omega_i\}) \right\} \\ &= \sum_{i=1}^{N} X(\omega_i)P(\{\omega_i\}) + \sum_{i=1}^{N} Y(\omega_i)P(\{\omega_i\}) \\ &= E[X] + E[Y]. \end{aligned}$$

□

2. Monotonicity

Let X and Y be two random variables. If $X(\omega) \le Y(\omega)$ for any $\omega \in \Omega$, then $E[X] \le E[Y]$.

Proof.

$$E[X] = \sum_{i=1}^{N} X(\omega_i)P(\{\omega_i\}) \le \sum_{i=1}^{N} Y(\omega_i)P(\{\omega_i\}) = E[Y].$$

Recall that each $P(\{\omega_i\})$ is positive. $\square$

3. Constants preserved
Let X be a random variable defined as $X(\omega) = c \in \mathbf{R}$ for any $\omega \in \Omega$. We have then $E[X] = c$.
Proof.

$$E[X] = \sum_{i=1}^{N} X(\omega_i)P(\{\omega_i\}) = \sum_{i=1}^{N} cP(\{\omega_i\}) = c.$$

$\square$

Let f be a function on $\mathbf{R}$. We define the expectation of $f(X)$ as

$$E[f(X)] := \sum_{i=1}^{N} f(X(\omega_i))P(\{\omega_i\}).$$

For example, we have

$$E[X^2] = \sum_{i=1}^{N} X^2(\omega_i)P(\{\omega_i\}),$$

and, if $f(x) = x^2 + 4x + 3$, then the linearlity and the constants preserved imply

$$E[f(X)] = E[X^2 + 4X + 3] = E[X^2] + 4E[X] + 3.$$

Variance
Let X be a random variable. $X - E[X]$ itself is also a random variable whose expectation is given by 0. Moreover, $(X - E[X])^2$ is a nonnegative random variable. Roughly speaking, when $(X - E[X])^2$ is large value, X derivates greatly from $E[X]$. When small, X is near to $E[X]$. Thus, $E[(X - E[X])^2]$ would be a useful measure of how much X tends to vary from $E[X]$. Now, we define the variance of X as

$$Var[X] := E[(X - E[X])^2].$$

Remark that

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] = E[X^2 - 2XE[X] + \{E[X]\}^2] \\ &= E[X^2] - 2E[XE[X]] + \{E[X]\}^2 = E[X^2] - \{E[X]\}^2. \end{aligned}$$

Example 1 (Rolling a die)

$$\begin{aligned} Var[Y] &= E[(Y - E[Y])^2] \\ &= \sum_{i=1}^{6} \left(Y(\omega_i) - \frac{7}{2} \right)^2 P(\{\omega_i\}) \\ &= \sum_{i=1}^{6} \left(i - \frac{7}{2} \right)^2 \frac{1}{6} = \frac{35}{12}. \end{aligned}$$

Covariance, Standard deviation and Correlation coefficient
Considering two random variables, these covariance refers to these mutual dependence. Let X and Y be two random variables. The covariance of X and Y is defined as

$$Cov(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Note that we have $Cov(X, Y) = E[XY] - E[X]E[Y]$.

The standard deviation $\sigma(X)$ of X is defined as follows:

$$\sigma(X) := \sqrt{Var[X]}.$$

In addition, the definition of the correlation coeficient ρ_{XY} of X and Y is given by

$$\rho_{XY} := \frac{Cov(X, Y)}{\sigma(X)\sigma(Y)}.$$

Note that $-1 \leq \rho_{XY} \leq 1$ by the Cauchy-Schwarz inequality.

Independence
Let A and B be events, that is, subsets of Ω. A and B are said to be indenpendent if $P(A \cap B) = P(A)P(B)$. Furthermore, for two random variables X and Y, these are said to be independent if

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\})P(\{Y = y\})$$

for any $x, y \in \mathbf{R}$. Recall that $\{X = x\} = \{\omega | X(\omega) = x\}$.