Conducting Comparative Statics when Functional Forms are General

Supplementary Note for Open Economy Macroeconomics Lecture Notes I

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In the "Open Economy Macroeconomics Lecture Notes 1", functional forms in the IS-LM model are not specified. A function whose form is not specified, such as $I(r)$ or $L(Y, r)$ has a "general functional form". In this note, we briefly review how to conduct comparative statistics when functional forms in a model are general.

The IS curve represents a set of combinations of (Y, r) that satisfies the equilibrium condition in the goods and services market.[1] The equilibrium condition, or the IS equation (see page I-1), is written as

$$Y = C(Y - T(Y)) + I(r) + G. \tag{1}$$

Why do we have to totally differentiate the equation (1) in obtaining a slope of the IS curve or in conducting comparative statics using the IS-LM equations? Why can't we just use partial differentiation instead of the total differentiation, and obtain $\partial r/\partial Y$, the slope of the IS curve? These questions relate to the functional forms of in the equation (1).

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[1]IS curve $= \{(Y, r) \,|\, Y = C(Y - T(Y)) + I(r) + G,\, dG = 0\}$

1 Specific vs. General Functional Forms

1.1 Specific Functional Forms

Suppose the equilibrium condition of the goods and services market and other functions are written as

$$
\begin{aligned}
Y &= C + I + G \\
C &= c_0 + a(Y - T) \qquad (c_0 > 0,\ 0 < a < 1) \\
T &= bY \qquad (0 < b < 1) \\
I &= c + dr \qquad (c > 0,\ d < 0).
\end{aligned}
$$

This system of equations explicitly specifies the functional forms of C, T, and I. Therefore we can rewrite the IS equation $Y = C + I + G$ in a reduced form $r = \frac{Y(1-a(1-b))-c_0-c-G}{d}$, of which the right-hand side consists of only exogenous variables and parameters.[2] In this case, we can partially differentiate r with respect to Y and obtain the slope of the IS curve, which is $\partial r/\partial Y = \frac{1-a(1-b)}{d} < 0$.

1.2 General Functional Forms

In contrast, the equation (1) contains functional forms that are not specified, meaning that we do not know the specific relationship between C and $(Y - T(Y))$ or I and r. In this case, we can neither rewrite the equation in a reduced form of "$r =$" or "$Y =$", nor apply partial differentiation in obtaining the slope of the IS curve.[3] When functional forms are not specified, we call them "general functional forms."

Moreover, in the IS equation (1), the variable C is dependent on Y. This causes another problem in obtaining a slope of the IS curve using partial differentiation. Suppose, hypothetically, that C is independent from Y, and rewrite the equation (1) as

$$Y = C + I(r) + G. \tag{2}$$

[2] Although both Y and r are the endogenous variables in the IS-LM analysis, we pay attention only to the IS equation in this section. That is why we call $r = \frac{Y(1-a(1-b))-c_0-c-G}{d}$ a reduced form.

[3] Strictly speaking, we do not even know either if the equation (1) can "define" a function $r = f(Y)$, or if r can be differentiated with respect to Y. The only thing we know from equation (1) is that we have a function $F(r, Y) = 0$. In order to verify if we can define a function $r = f(Y)$ from the $F(r, Y)$ and differentiate r with respect to Y, we have to use the Implicit Function Theorem. The theorem states that we have a unique continuously differentiable function $f : Y \longrightarrow r$ near a point (r_0, Y_0) if and only if (a) the $F(r, Y)$ is continuously differentiable, and (b) partial derivative of F with respect to r is non-zero at the point (r_0, Y_0) that satisfies $F(r, Y) = 0$. In the case of the equation (1), if we suppose that the $F(r, Y)$ is continuously differentiable, then the theorem holds because $F_Y = -I_r > 0$. This at least implies that a slope of the IS curve (dr/dY) is not vertical.

We can apply partial differentiation to equation (2) even if it contains variable with a general functional form $I(r)$, and obtain $\partial Y/\partial r = \partial I/\partial r$. Since $\partial Y/\partial r = \partial I/\partial r < 0$, the slope of the IS curve $(\partial r/\partial Y)$ corresponding to the equation (2) should also be negative $(\partial r/\partial Y = \partial r/\partial I < 0)$ in order for the equilibrium condition to be maintained.

However, as previously mentioned, the variable C in equation (1) is dependent on the endogenous variable Y. This means that while a change in r directly affects Y, this impact of r on Y *per se* affects Y again via $C(Y - T(Y))$. The right-hand side in a reduced form $(Y =)$ must consist of only exogenous variables and parameters. If it is not a reduced form, we can not apply partial differentiation to the equation as it ignores the indirect (second) impact of r on Y. We therefore must use total differentiation in obtaining dr/dY, the slope of IS curve.

2 Some Mathematical Review

2.1 Review of Total Differentiation

The general expression for total differentiation of a function $y = f(x, z)$ is written as

$$\begin{aligned} dy &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial z}dz \\ &= f_x dx + f_z dz \end{aligned} \tag{3}$$

where $\partial f/\partial x \equiv f_x$, $\partial f/\partial z \equiv f_z$. The derivative f_x measures how dx (change in x) affects y, and the derivative f_z measures how dz (change in z) affects y. Therefore, dy is the sum of the impact of a change in x on y and that of a change in z on y.

In the case of the relationship between (x, z and y), it can not be drawn in (x, y) two-dimensional plane as it consists of three variables. However, if you treat z as a constant, the xy curve (i.e. a set of combinations of (x, y) that satisfies the $y = f(x, z)$) can be drawn in the (x, y) two-dimensional plane as one surface of the function.[4]

Next, in order to find a slope of the xy curve when z is constant $(dz = 0)$, just divide both sides of the equation (3) by dx.

$$\left.\frac{dy}{dx}\right|_{xy,\, dz=0} = f_x + f_z dz = f_x + 0 = f_z. \tag{4}$$

[4] xy curve $= \{(x, y) \,|\, y = f(x, z),\, dz = 0\}$

2.2 Review of the Chain Rule

The general form of the Chain Rule, the rule for differentiating a composite function, is written as

$$\frac{d}{dx}(h(g(x))) = h'(g(x))g'(x) \tag{5}$$

Suppose a consumption function is written as $C = C(Y - T(Y))$. This is a composite function and we can apply the chain rule in differentiating C with respect to Y.

$$\begin{aligned}
\frac{dC}{dY} &= \frac{\partial C}{\partial(Y - T(Y))}\frac{\partial(Y - T(Y))}{\partial Y} \\
&= \frac{\partial C}{\partial(Y - T(Y))}\left(1 - \frac{\partial T}{\partial Y}\right) \\
&= \frac{\partial C}{\partial Y^D}\left(1 - \frac{\partial T}{\partial Y}\right).
\end{aligned}$$

3 Obtaining the Slope of the IS-LM Curve

3.1 The IS Curve

By applying the chain rule and total differentiation to equation (1), the slope of the IS curve (dr/dY) in the (r, Y) plane can be obtained.

$$\begin{aligned}
Y &= C(Y - T(Y)) + I(r) + G \\
dY &= \frac{\partial C}{\partial(Y - T(Y))}\frac{\partial(Y - T(Y))}{\partial Y}dY + \frac{\partial I}{\partial r}dr + dG \\
&= \frac{\partial C}{\partial Y^D}\left(1 - \frac{\partial T}{\partial Y}\right)dY + I_r dr + dG \\
&= C_{Y^D}(1 - T_Y)dY + I_r dr + dG, \qquad (6)
\end{aligned}$$

where $Y^D \equiv Y - T(Y)$ and $C_{Y^D} \equiv \frac{\partial C}{\partial Y^D}$. As previously mentioned, in the case of the relationship between (Y, G and r), it can not be expressed in the (r, Y) two-dimensional plane. We therefore treat G as given in obtaining the slope of the IS equation as (r, Y) are endogenous variables and G is treated as exogenous in the IS-LM analysis. From the equation (6), we have

$$\begin{aligned}
[1 - C_{Y^D}(1 - T_Y)]dY &= I_r dr + dG \qquad (7) \\
\left.\frac{dr}{dY}\right|_{IS,\, dG=0} &= \frac{1 - C_{Y^D}(1 - T_Y)}{I_r} < 0.
\end{aligned}$$

Notice that in the Lecture Notes, dC/dY^D is denoted C_Y, not C_{Y^D} (see page I-2).

3.2 The LM Curve

The LM curve represents a set of combinations of (Y, r) that satisfies the equilibrium condition in the money market. The equilibrium condition, or the LM equation, is written as[5]

$$\frac{M}{P} = L(Y, r). \tag{8}$$

Again, by totally differentiating equation (8), the slope of the LM curve (dr/dY) in the (r, Y) plane can be obtained.

$$\begin{aligned} \frac{\partial(\frac{M}{P})}{\partial M}dM + \frac{\partial(\frac{M}{P})}{\partial P}dP &= \frac{\partial L(\cdot)}{\partial Y}dY + \frac{\partial L(\cdot)}{\partial r}dr \\ \frac{1}{P}dM - \frac{M}{P^2}dP &= L_Y dY + L_r dr, \end{aligned} \tag{9}$$

where $L_Y \equiv \frac{\partial L(\cdot)}{\partial Y}$ and $L_r \equiv \frac{\partial L(\cdot)}{\partial r}$.[6] Again, in the case of the relationship between (Y, M, P) and r, it can not be expressed in (r, Y) two-dimensional plane. We therefore treat M as well as P as given in obtaining the slope of the LM equation, because (Y, r) are endogenous and (M, P) are exogenous in the IS-LM analysis. From the equation (9), we have

$$\left.\frac{dr}{dY}\right|_{LM,\, dM=0,\, dP=0} = -\frac{L_r}{L_Y} > 0.$$

4 The IS-LM Analysis

4.1 Specific Functional Forms

Again, if we had IS and LM equations with specific functional forms, we could rewrite them as two functions of $r = f(Y)$, for instance,

$$r = \frac{Y(1 - a(1 - b)) - c_0 - c - G}{d}$$

for the IS equation and

$$r = \frac{eY - \frac{M}{P}}{f}$$

for the LM equation, and substitute out r or Y to make a reduced form. With this reduced form, we could apply partial differentiation to Y or r with respect to G or M to conduct comparative statics (IS-LM analysis). However, in the Lecture Notes, we only have equations with general functional forms. We therefore must find another way to conduct the IS-LM analysis.

[5] LM curve $= \left\{(Y, r) \mid \frac{M}{P} = L(Y, r),\ dM = 0,\ dP = 0\right\}$

[6] Recall that $\frac{d(\frac{M}{P})}{dP} = \frac{d(MP^{-1})}{dP} = -MP^{-2} = -\frac{M}{P^2}$.

4.2 General Functional Forms

4.2.1 Basic Concept

The fact that we can not rewrite the two equations (1) and (8) as reduced forms of $r = f(Y)$ means that we can never obtain solutions (equilibrium values) for r or Y. However, our original goal is not to find a value for r or Y, but to obtain the derivatives dY/dG, dr/dG, dY/dM, dr/dM. This means that we do not have to solve equations for Y or r. All we do is to solve a very simple system of equations for the derivatives dY/dG, dr/dG, dY/dM, dr/dM. From (7) and (9)

$$[1 - C_{Y^D}(1 - T_Y)]dY = I_r dr + dG \tag{10}$$

$$L_Y dY + L_r dr = \frac{1}{P}dM - \frac{M}{P^2}dP. \tag{11}$$

In the case of obtaining dY/dG or dr/dG, we treat M as given ($dM = 0$). Therefore, the system can be rewritten as,

$$[1 - C_{Y^D}(1 - T_Y)]dY - I_r dr = dG$$

$$L_Y dY + L_r dr = 0,$$

and

$$[1 - C_{Y^D}(1 - T_Y)]\frac{dY}{dG} - I_r\frac{dr}{dG} = 1 \tag{12}$$

$$L_Y\frac{dY}{dG} + L_r\frac{dr}{dG} = 0. \tag{13}$$

This is the simultaneous equation that we are going to solve in investigating the impact of dG on dY or dr. How to solve this system of equations basically depends on your preference. If the number of endogenous variables is small, you can even solve it by substituting out $\frac{dr}{dG}$ or $\frac{dY}{dG}$, just as solving for x and y in the simplest system of equations

$$ax + by = e \tag{14}$$

$$cx + dy = f. \tag{15}$$

However, if the number of equations (or endogenous variables) is large, matrix operation would be more comvenient.

4.2.2 Solvability

By the way, the solutions for (14) and (15) is $x = \frac{de-bf}{ad-bc}$ and $y = \frac{cd-af}{bc-ad}$. Therefore, this system of equations has solutions if and only if $ad - bc \neq 0 \iff ad \neq bc$, as the denominator can not be zero. What does this mean intuitively?

Suppose $b = 1$ and $d = 1$, then the equations become $ax + y = e$ and $cx + y = f$. Then the condition $ad - bc \neq 0$ becomes $a - c \neq 0 \Longleftrightarrow a \neq d$. This implies when $a = d$, the two lines do not intersect each other in (x, y) plane (e.g., $y = 2x + 3$ and $y = 2x + 5$) as the slopes are the same, which means we do not have any solution for x or y. The condition $ad - bc \neq 0$ also ensures that the two equations are not linearly dependent (e.g., $y = 2x + 3$ and $2y = 4x + 6$), which means the two lines are identical.

4.2.3 Determinant

In the case of the matrix operation $\boldsymbol{Ax} = \boldsymbol{d}$, the "determinant" of a coefficient matrix $\boldsymbol{A}$ plays exactly the same role as the "$ad - bc$" in equations (14) and (15). Actually, $|\boldsymbol{A}| \equiv ad - bc$ in a 2×2 coefficient matrix, and $|\boldsymbol{A}| \neq 0$ ensures that the equations are solvable. This can be reconfirmed by recalling the inverse matrix formula $\boldsymbol{A}^{\boldsymbol{-1}}$ or the Cramer's Rule formula, in which $|\boldsymbol{A}|$ always appears as the denominator, and this condition holds even in the case of solving n equations with general functional forms.[7] Although you can also find values for $\boldsymbol{x}$ by calculating an inverse matrix, using Cramer's Rule would be a less time-consuming way in conducting comparative statics.

[7] Also review the Implicit Function Theorem in the case of a system of equations.